Power Series

Anton 11.8

CENTERED AT $x=0$

Power Series in *x*: a series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Examples:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$ GEOMETRIC, $r = x$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

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We would like to find for what values of x the series converges. What test should we use?

Ratio Test for Absolute Convergence

Let $\rho = \lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right|$

1. If $\rho < 1$ then the series converges absolutely.
2. If $\rho > 1$ then the series diverges.
3. If $\rho = 1$ then the test is inconclusive*.

*Further examination is required.

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Example: Find the interval and radius of convergence. USE RATIO TEST

NEED $\rho < 1$ TO CONVERGE

$$\sum_{k=0}^{\infty} x^k \Rightarrow \rho = \lim_{k\to\infty}\left|\frac{x^{k+1}}{x^k}\right| = |x| < 1$$

$\therefore |x| < 1 \Rightarrow \boxed{-1 < x < 1}$

IF $x = \pm 1$, $\rho = 1 \Rightarrow$ FURTHER EXAMINATION

CHECK ENDPOINTS:

$x = -1$: $\sum_{k=0}^{\infty} (-1)^k = 1-1+1-1+\dots$ DIVERGES

$x = 1$: $\sum_{k=0}^{\infty} (1)^k = 1+1+1+\dots$ DIVERGES

INTERVAL OF CONVERGENCE: $-1 < x < 1$, OR $(-1, 1)$, OR $|x| < 1$

RADIUS OF CONVERGENCE: $\underline{1}$

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Example: Find the interval and radius of convergence.

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \Rightarrow \rho = \lim_{k\to\infty} \left| \frac{\cancel{x^{k+1}}}{(k+1)\cancel{k!}} \cdot \frac{\cancel{k!}}{\cancel{x^k}} \right| = \left| \frac{x}{k+1} \right| = 0 < 1$$

$$\therefore \sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ CONV. FOR ALL } x \Rightarrow \text{INTERVAL} = (-\infty, \infty)$$

RADIUS OF CONV $= \infty$

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Example: Find the interval and radius of convergence.

$$\sum_{k=0}^{\infty} k!x^k \Rightarrow \rho = \lim_{k\to\infty} \left| \frac{(k+1)\cancel{k!}\,x^{\cancel{k+1}}}{\cancel{k!}\,\cancel{x^k}} \right| = \left| (k+1)\cdot x \right| \to \infty > 1$$

$$\therefore \sum_{k=0}^{\infty} k!x^k \text{ DIVERGES FOR } x \neq 0$$

OR, CONVERGES FOR $x = 0$. (RADIUS $= 0$)

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Example: Find the interval and radius of convergence.

$$\sum_{k=0}^{\infty}(-1)^k \frac{x^k}{3^k(k+1)} \Rightarrow \rho = \lim_{k\to\infty}\left|\frac{x^{k+1}}{3^{k+1}(k+2)}\cdot\frac{3^k(k+1)}{x^k}\right| = \left|\frac{x}{3}\right| < 1$$

$$\left|\frac{x}{3}\right| < 1 \Rightarrow -1 < \frac{x}{3} < 1 \Rightarrow -3 < x < 3$$

CHECK ENDPTS:

$x=-3$: $\sum \frac{(-1)^k(-3)^k}{3^k(k+1)} = \sum \frac{1}{k+1} \rightarrow$ DIV. HARMONIC

$x=3$: $\sum \frac{(-1)^k(3)^k}{3^k(k+1)} = \sum \frac{(-1)^k}{k+1} \rightarrow$ CONV. ALT. HARMONIC

INTERVAL OF CONV: $-3 < x \le 3$ OR $(-3, 3]$

RADIUS OF CONV: 3

Convergence/Divergence of a Power Series in *x*: (CENTERED AT $x=0$)

Exactly one of the following is true:

1. Series converges only for $x = 0$.
2. Series converges absolutely for all x.
3. Series converges for x in an interval $(-R, R)$ and diverges everywhere else. At R or $-R$, the series may converge absolutely, conditionally, or it may diverge – you need to check each endpoint.

R is called the *radius of convergence*.

CENTERED AT $x = a$

Power Series in $(x - a)$: a series of the form

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Example:

$$\sum_{k=\cancel{0}\,1}^{\infty} \frac{(x-5)^k}{k^2} = (x-5) + \frac{(x-5)^2}{2^2} + \frac{(x-5)^3}{3^2} + \cdots$$

Find the interval and radius of convergence.

$$\sum_{k=\cancel{0}\,1}^{\infty} \frac{(x-5)^k}{k^2} \Rightarrow \rho = \lim_{k\to\infty} \left| \frac{(x-5)^{\cancel{k+1}}}{(k+1)^2} \cdot \frac{k^2}{\cancel{(x-5)^k}} \right| = |x-5| < 1$$

$$|x-5| < 1 \Rightarrow -1 < x-5 < 1 \Rightarrow \boxed{4 < x < 6}$$

CHECK ENDPTS

$x = 4$: $\sum \frac{(-1)^k}{k^2} \rightarrow$ CONV. ALT P-SERIES $p = 2$

$x = 6$: $\sum \frac{1}{k^2} \rightarrow$ CONV. P-SERIES $p = 2$

INT. OF CONV: $4 \le x \le 6$ OR $[4, 6]$

RADIUS = 1

Convergence/Divergence of a Power Series in (x – a):

Exactly one of the following is true:

1. Series converges only for $x = a$.
2. Series converges absolutely for all x.
3. Series converges for x in an interval $(a–R, a+R)$ and diverges everywhere else. The series may converge absolutely, conditionally, or it may diverge at the endpoints of the interval – you need to check.

R is called the *radius of convergence*.

Homework:

Anton 11.8 # 1 – 23 odd